Collision of spinning particles near BTZ black holes*

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Abstract: We study the collision property of spinning particles near a Bañados-Teitelboim-Zanelli (BTZ) black hole. Our results show that although the center-of-mass energy of two ingoing particles diverges if one of the particles possesses a critical angular momentum, however, particle with critical angular momentum can not exist outside of the horizon due to the violation of timelike constraint. Further detailed investigation indicates that only a particle with a subcritical angular momentum is allowed to exist near an extremal rotating BTZ black hole and the corresponding collision center-of-mass energy can be arbitrarily large in a critical angular momentum limit.

1 Introduction

The particle collision near a black hole background has long history. The possibility of having infinite center-of-mass energy collision near a black hole was first pointed out by Piran, Shaham, and Katz in 1975[1]. In 2009, Bañados, Silk, and West[2], rediscovered this mechanism, known as the BSW process, and they pointed out that because of infinite center-of-mass energy caused by collision, the rotating black holes can thus act as particle accelerators[2, 3]. Along this line, many aspects of BSW mechanism with various black hole backgrounds have been investigated. For instance, the Kerr naked singularity[4], the charged spinning black hole[5], the Kerr-(anti)de-Sitter black hole spacetime [6] and the universal property of rotating black holes was given in [7]. Other research related to higher or lower dimensional spacetime background [8–10] is also interesting, such as five dimensional Kerr black hole can be found in[8] and three dimensional rotating charged hairy black hole have been studied in [10]. Furthermore, the BSW mechanism can help us to optimize the collisional penrose process which extracts energy from a black hole through particle collision[11–19].

In three dimensional spacetime, there exists a typical stationary black hole solution with a negative cosmological constant which was first discovered by Bañados-Teitelboim-Zanelli (BTZ)[20]. This black hole solution, because of its similarity and simplicity compared with the (3+1)-dimensional Kerr black hole, has received increasing attention recently. For example, the spinless particles

collision around BTZ black hole has been study in [21]. People are interested in the (2+1)-dimensional BTZ black hole because it can be a good toy model which help to gain more deep understanding of the same problem in the Kerr spacetime, since in the BTZ background the analytical expression usually is possible[22–24], while in the Kerr spacetime the same analytical treatment for the same problem is generally very difficult. For example, the collision of fast rotating dust thin shells in (2+1)-dimension is much simpler compared with the (3+1)-dimensional Kerr spacetime[22–25].

On the other hand, many authors focus on point particle whose trajectory is a geodesic. However a real particle should be an extended body with inclusion of self-interaction. Compared with the spinless particle, the orbit of a spinning test particle is no longer a geodesic, and it has been shown that[26–32] the equations of motion of spinning particles around a given spacetime background is discribed by the Mathisson-Papapetrou-Dixon(MPD) equations [33–35]. By collecting these results, the authors in [30] show that the collision center-of-mass energy could be divergent for extremal Kerr black hole. With these motivations, our research in this paper is devoted to study the collision of spinning particles around the BTZ black hole.

The paper is organized as follows: In section II, we introduce Mathission-Papapetrou-Dixon(MPD) equations, which describes the spinning particles' motion in curved spacetime, and apply it to Bañados-Teitelboim-Zanelli(BTZ) black hole. In section III, we obtain the collision center-of-mass energy of spinning particles and

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find the condition for the divergence of center-of-mass energy with either of the particle possessing critical total angular momentum. Then in section IV, the motion of spinning particles with critical and subcritical angular momentum near the event horizon are analyzed in details, and it's shown a spinning particle with subcritical total angular momentum are allowed to exist on or outside the horizon. Next, in section V, collision of two spinning particles with subcritical total angular momenta near the horizon are calculated, and the diverging center-of-mass energy in the critical limit is obtained. Conclusions are given in the last section VI. Through out the paper, we adopt the convention that the speed of light $c = 1$.

2   Equations of motion for spinning particles

The the metric of BTZ black hole in the Boyer-Lindquist coordinates reads[20]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\phi - \frac{r_+r_-}{lr^2}dt\right)^2, \quad (1)$$

where

$$g(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2r^2}, \quad (2)$$

and

$$M = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad (3)$$

$$J = \frac{r_+r_-}{4Gl}. \quad (4)$$

and $r = r_+$ is the outer horizon, $r = r_-$ is the inner horizon, M is the ADM mass, J is the angular momentum and l is a parameter determined by the negative cosmological constant Λ ($l^2 = -\Lambda/3$). Note that for the angular momentum J, $|J| \leq Ml$ must be satisfied. When the black hole is extremal($r_+ = r_-$), we have $|J| = Ml$.

Under the given BTZ spacetime, the spinning particle's motion can be described by MPD equations[28, 30]

$$\frac{D}{D\tau}P^a = -\frac{1}{2}R^a_{bcd}v^bS^{cd}, \quad (5)$$

$$\frac{D}{D\tau}S^{ab} = P^av^b - P^bv^a, \quad (6)$$

Along the center-of-mass world line $z(\tau)$, $v^a = (\frac{\partial}{\partial\tau})^a$ is the tangent vector, $\frac{D}{D\tau}$ is the covariant derivative, P^a is the momentum of the spinning particles, and S^{ab} is the spinning angular momentum tensor.

In order to obtain the detailed relation between P^a and v^a, supplementary conditions are required to be imposed: [31, 30]

$$S^{ab}P_b = 0, \quad (7)$$

$$P^av_a = -m, \quad (8)$$

where τ is not necessarily the proper time of the spinning particle. Combining the Eqs. (2.6), (2.8) and (2.9), the difference between v^a and u^a reads[30, 34]

$$mv^a - P^a = \frac{S^{ab}R_{bcde}P^cS^{de}}{2\left(m^2 + \frac{1}{4}R_{bcde}S^{bc}S^{de}\right)}. \quad (9)$$

With direct calculation, we find that $v^a = u^a$ in BTZ spacetime where $u^a \equiv P^a/m$. It should be noted that the velocity v^a is parallel to the momentum u^a in the specific property in (2+1)-dimensions, and of course in generally not valid in 4-dimensional spacetime.

Note that, there are two Killing vector fields $\xi^a = (\partial/\partial t)^a$ and $\phi^a = (\partial/\partial\phi)^a$ in BTZ spacetime and because BTZ spacetime is axi-symmetric and stationary, they can be expanded in the orthonormal triad basis $e_a^{(\nu)}$ as

$$\xi_a = -\sqrt{g(r)}e_a^{(0)} - \frac{r_+r_-}{lr}e_a^{(2)},$$
$$\phi_a = re_a^{(2)}. \quad (10)$$

where

$$e_a^{(0)} = \sqrt{g(r)}(dt)_a,$$
$$e_a^{(1)} = \frac{1}{\sqrt{g(r)}}(dr)_a, \quad (11)$$
$$e_a^{(2)} = r\left((d\phi)_a - \frac{r_+r_-}{lr^2}(dt)_a\right).$$

Then, a corresponding conserved quantity can be defined by Killing vector field ξ^a as follows:

$$Q_\xi = P^a\xi_a - \frac{1}{2}S^{ab}\nabla_b\xi_a. \quad (12)$$

From the equation above, two conserved quantities can be obtained, namely the energy of per unit mass of the particle E_m, and the angular momentum per unit mass of the particle J_m, they are

$$E_m = -u^a\xi_a + \frac{1}{2m}S^{ab}\nabla_b\xi_a,$$
$$J_m = u^a\phi_a - \frac{1}{2m}S^{ab}\nabla_b\phi_a. \quad (13)$$

Combining with Eqs.(2.7) and (2.8) we can introduce the spin s of the particle as

$$s^2 := \frac{1}{2m^2}S^{ab}S_{ab}, \quad (14)$$

where s is the spin of unit mass. What's more, Combining with Eq.(2.6), (2.7) and (2.8), the spin tensor can be written reversely as

$$S^{(a)(b)} = -m\varepsilon^{(a)(b)}_{\quad(c)}u^{(c)}s, \quad (15)$$

where $\varepsilon_{(a)(b)(c)}$ is the completely anti-symmetric tensor with the component $\varepsilon_{(a)(b)(c)} = 1$.

From Eq.(2.16), the non-zero components of the spin tensor can be expressed in terms of $u^{(a)}$ as

$$S^{(0)(1)} = -msu^{(2)},$$
$$S^{(0)(2)} = msu^{(1)},$$
$$S^{(1)(2)} = -msu^{(0)}. \qquad (16)$$

The explicit expressions of the energy and the angular momentum per unit mass E_m and J_m in terms of $u^{(a)}$ can be obtained by using Eq. (2.17) and Eq. (2.14) as:

$$E_m = \sqrt{g(r)}u^{(0)} + \left(\frac{r_+r_-}{lr} + \frac{rs}{l^2}\right)u^{(2)}, \qquad (17)$$

$$J_m = s\sqrt{g(r)}u^{(0)} + \left(\frac{r_+r_- s}{lr} + r\right)u^{(2)}. \qquad (18)$$

Solving Eq. (2.18) and Eq. (2.19) gives

$$u^{(0)} = \frac{l\left(lE_m\left(lr^2 + r_-r_+s\right) - J_m\left(lr_-r_+ + r^2s\right)\right)}{r\sqrt{\left(r^2 - r_-^2\right)\left(r^2 - r_+^2\right)}(l^2 - s^2)}, \qquad (19)$$

$$u^{(2)} = \frac{J_m - E_m s}{r - \frac{rs^2}{l^2}}. \qquad (20)$$

By considering the normalization condition of momentum $u^{(a)}u_{(a)} = -m^2$, we obtain the $u^{(1)}$ as follow:

$$(u^{(1)})^2 = (u^{(0)})^2 - (u^{(2)})^2 - m^2. \qquad (21)$$

For direct comparison to the spinless case in [21], now we express the momentum in coordinate basis:

$$p^t(r) = \frac{dt}{d\tau} = \frac{W(r)}{g(r)}, \qquad (22)$$

$$p^r(r) = \frac{dr}{d\tau} = \rho\sqrt{Y(r)}, \qquad (23)$$

$$p^\phi(r) = \frac{d\phi}{d\tau} = \frac{r_+r_- W(r)}{lg(r)r^2} + \frac{l^2(J_m - E_m s)}{r^2(l^2 - s^2)}. \qquad (24)$$

where

$$W(r) = \frac{E_m l\left(lr^2 + r_-r_+s\right) - J_m\left(lr_-r_+ + r^2s\right)}{r^2\left(l^2 - s^2\right)}, \qquad (25)$$

$$Y(r) = W^2(r) - \left(m^2 + \left(\frac{J_m - E_m s}{r(1 - \frac{s^2}{l^2})}\right)^2\right)g(r), \qquad (26)$$

$\rho = +1$ for outwards direction, -1 for inwards direction.

We define the critical angular momentum as

$$J_c \equiv \frac{E_m l(lr_+ + r_-s)}{lr_- + r_+ s}, \qquad (27)$$

and a particle with critical angular momentum J_c corresponds to:

$$W_i(r_+) = 0, \qquad (28)$$

where $i = 1, 2$ refers to particle 1 or particle 2 in the collision process. When the particle's spin $s = 0$, the critical angular momentum introduced here will be reduced to the spinless case, which already has been investigated in [21].

The timelike constraint of Eq. (2.23) means $p^t(r) > 0$

outside the horizon for massive particles, which in turn implies $W_i(r) > 0$. Therefore, for particles with the angular momentum $J_m \leqslant J_c$, the positivity of $W_i(r)$ gives rise to a constraint on particle's spin as $l^2 - s^2 > 0$. Therefore in the following sections, we restrict ourselves to the case $-l < s < l$.

3   Center-of-mass energy of the collision

In this section we intend to find the condition required for infinite center-of-mass energy collision of two spinning massive particles near the BTZ horizon. They are particle $i = 1, 2$ that starts at infinity with masses m_i, energy per unit mass E_{mi}, total angular momenta per unit mass J_{mi}, and spins s_i respectively, falling to the black hole and colliding near the event horizon. Then the collision center-of-mass energy E_{cm} is given by[21, 30]:

$$E_{cm}^2 \equiv -(p_1^\mu(r) + p_2^\mu(r))(p_{1\mu}(r) + p_{2\mu}(r)) = m_1^2 + m_2^2$$
$$+ \frac{W_1(r)W_2(r) - \sqrt{Y_1(r)Y_2(r)}}{g(r)}$$
$$- 2\frac{l^4(J_{m1} - E_{m1}s_1)(J_{m2} - E_{m1}s_2)}{r^2(l^2 - s_1^2)(l^2 - s_2^2)}, \qquad (29)$$

where $Y_i(r)$ and $W_i(r)$ are defined by Eq. (2.26) and (2.27) with $i = 1, 2$ again labeling particle 1 or particle 2.

We find that the third term of Eq. (3.1) is a $\frac{0}{0}$ type when r approaches to the event horizon r_+, so we first need to regularize this term as

$$\lim_{r\to r_+} 2\frac{W_1(r)W_2(r) - \sqrt{Y_1(r)Y_2(r)}}{g(r)} = \frac{W_2(r_+)}{W_1(r_+)}Z_1 + \frac{W_1(r_+)}{W_2(r_+)}Z_2, \qquad (30)$$

in which $Z_i = \left(m_i^2 + \left(\frac{J_{mi} - E_{mi}s_i}{r(1 - \frac{s_i^2}{l^2})}\right)^2\right) > 0. \qquad (31)$

It is easy to see that E_{cm}^2 blows up with $r \to r_+$ if one of the particle has the critical angular momentum J_c(which means $W_i(r_+) = 0$). If both particles possess J_c, then we have,

$$\frac{W_2(r_+)}{W_1(r_+)} = \frac{W_2'(r_+)}{W_1'(r_+)} = \frac{E_{m1}(lr_- + r_+s_2)}{E_{m2}(lr_- + r_+s_1)}, \qquad (32)$$

in which $\prime$ denotes the derivative with respect to r. For equal spin collision ($s_1 = s_2$), the ratio $\frac{W_2(r_+)}{W_1(r_+)} = \frac{E_{m1}}{E_{m2}}$ becomes finite value, which is similar to the spinless case [21]. Therefore, the only possibility for the center-of-mass energy goes to infinity is one of the spin, for example, s_1, satisfies

$$s_1 = s_c = -\frac{lr_-}{r_+}. \qquad (33)$$

However, this is equivalent to require $J_{m1} = J_{c1}$ to be infinity according to Eq. (2.28) and thus is impossible to

achieve in practice.

4   Motion of a particle with critical and subcritical total angular momentum

In last section, we showed that if one of the collision particle possesses critical angular momentum, the center-of-mass energy E_{cm} will blow-up. However, to solid this conclusion, we still need to check whether the particle with critical angular momentum J_c can satisfy other constraints such as timelike constraint in subsection IVA and radial equation of motion which guarantees the particles can reach the horizon. Therefore, the aim of this section is to discuss these constraints carefully.

First we note that for spinless case[21], a particle with critical total angular momentum $J_m = J_c$ is not allowed to exist outside the event horizon while one with subcritical angular momentum can be allowed. Later we shall investigate the same issue by taking account of spin effect in subsections IVB to IVC.

4.1   Timelike constraint of $p^t(r)$

The first subsection is devoted to the timelike constraint of $p^t(r)$. To avoid superluminality, $p^t(r)$ should be non-negative, from Eq. (2.23) we have,

$$p^t(r) = \frac{W(r)}{g(r)} \geqslant 0, \tag{34}$$

since $f(r) > 0$, the above equation is equivalent to,

$$W(r) = \frac{l\left(lE_m\left(lr^2 + r_-r_+s\right) - J_m\left(lr_-r_+ + r^2s\right)\right)}{r(l^2 - s^2)} \geqslant 0. \tag{35}$$

Eq. (4.2) gives a restriction of J_m to ensure $p^t(r) \geqslant 0$ near the event horizon where the infinite center-of-mass energy collision takes place. Considering extremal black hole and the case $-l < s < l$, this gives rise to

$$J_m \leqslant E_m l = J_c, \tag{36}$$

which means for $J_m < J_c$, the timelike condition is satisfied. However, for massive particle, when the total angular momentum takes critical value $J_m = J_c$, the timelike condition is violated. Therefore, in the following sections, we will consider the subcritical total angular momentum $J_m < J_c$.

4.2   Radial motion of the particle: $Y(r)$

Now we come to the radial motion of the particle, we start with the expression of $p^r(r)$ (2.24), and obtain the radial equation of motion of the spinning particle:

$$\frac{1}{2}p^r(r)^2 + V(r) = 0, \tag{37}$$

where $V(r)$ is the radial effective potential defined by $V(r) \equiv -Y(r)/2$, and τ is the geodesic parameter. Particles are only allowed to exist in regions where $V(r) \leqslant 0$ or

$Y(r) \geqslant 0$ from Eq. (4.4).

For massive particle with $m \neq 0$ we consider, the tendency of $Y(r)$ at infinity is

$$\lim_{r \to \infty} Y(r) = -m^2 \times \infty < 0 \tag{38}$$

which implies a massive particle cannot escape to infinity. Since the expression of $Y(r)$ for massive particle is complicated, we will investigate it with subcritical total angular momentum in IVC, especially for extremal black hole.

4.3   Motion of a particle with the subcritical total angular momentum

In last subsection, we already know particles with critical total angular momentum cannot exist outside the event horizon. So we consider a particle with subcritical total angular momentum $J_m(J_m \leqslant J_c$ for the case $-l < s < l$ according to Eq. (4.3)):

$$J_m \equiv J_c - \tilde{\delta} = \frac{E_m l(lr_+ + r_-s)}{lr_- + r_+s} - \tilde{\delta}, \tag{39}$$

and try to find the range of $\tilde{\delta}$ that enables the particle to exist outside the black hole (i.e. $Y(r) \geqslant 0$). With this well-defined subcritical total angular momentum, we obtain the corresponding function $Y(r)$ as follows:

$$\begin{aligned} Y(r) = Y_c(r) + \big\{ \tilde{\delta}(-2E_m l(r^2 - r_+^2)(l^2 - s^2) \\ \times (lr_+ + r_-s) - (lr_- + r_+s) \\ (l^2(-r^2 + r_-^2 + r_+^2) + 2lr_-r_+s + r^2s^2)\tilde{\delta}) \big\} / \\ \left(r^2(l^2 - s^2)^2(lr_- + r_+s) \right). \end{aligned} \tag{40}$$

where

$$Y_c(r) = -\frac{(r^2 - r_+^2)}{r^2}\left(\frac{E_m^2 l^2(r_+^2 - r_-^2)}{(lr_- + r_+s)^2} + \frac{m^2(r^2 - r_-)^2}{l^2} \right). \tag{41}$$

When spin s is taken as zero, our result will reduce to the spinless case by identifying $\tilde{\delta} = \frac{r_+l}{r_-}\delta$ with δ introduced in [21]. From Eq. (4.7), a particle with subcritical total angular momentum can exist on the event horizon or nearby outside of the black hole since

$$Y(r_+) = \left(\frac{(lr_- + r_+s)\tilde{\delta}}{r_+(l^2 - s^2)} \right)^2 > 0. \tag{42}$$

Now we come to discuss $Y'(r)$, which is the derivative of $Y(r)$ with respect to r, determines how much the collision point departures from the event horizon r_+[21]. First, $Y'(r)$ with critical total angular momentum(i.e. $\tilde{\delta} = 0$) is

$$Y_c'(r_+) = \frac{2\left(r_-^2 - r_+^2\right)\left(l^4 E_m^2 + m^2\left(lr_- + r_+s\right)^2\right)}{l^2 r_+\left(lr_- + r_+s\right)^2} \tag{43}$$

on the event horizon. For the extremal case with critical total angular momentum, since we have $r_+ = r_-$, by using Eqs. (4.8) and (4.10), we obtain $Y(r_+) = Y'(r_+) = 0$ on the

event horizon.

Then with subcritical total angular momentum, $Y'(r)$ is relevant to $\tilde{\delta}$

$$Y'(r) = Y'_c(r) - \frac{2l\tilde{\delta}(-2E_m r_+^2(l^2 - s^2)(lr_+ + r_- s) + (lr_- + r_+ s)(l(r_-^2 + r_+^2) + 2r_- r_+ s)\tilde{\delta})}{r^3(lr_- + r_+ s)(l^2 - s^2)^2},$$ (44)

On the event horizon the above equation becomes

$$Y'(r_+) = D_2\tilde{\delta}^2 + D_1\tilde{\delta} + D_0,$$ (45)

where the coefficients read

$$D_2 = -\frac{2l\left(l\left(r_-^2 + r_+^2\right) + 2r_- r_+ s\right)}{r_+^3\left(l^2 - s^2\right)^2},$$ (46)

$$D_1 = \frac{4lE_m(lr_+ + r_- s)}{r_+(l^2 - s^2)(lr_- + r_+ s)},$$ (47)

$$D_0 = \frac{2\left(r_-^2 - r_+^2\right)\left(l^4 E_m^2 + m^2(lr_- + r_+ s)^2\right)}{l^2 r_+(lr_- + r_+ s)^2}.$$ (48)

The coefficient $D_2 < 0$ by considering $s^2 < l^2$, the sign of D_2 is the same as in spinless case[21]. Along the same line of [21], by solving (4.12), we found there exists a $E_{\max}$

$$E_{\max} = \frac{m\sqrt{r_+^2 - r_-^2}\sqrt{lr_-^2 + lr_+^2 + 2r_+ r_- s}}{l^{3/2} r_-},$$ (49)

if the unit mass energy $E_m < E_{\max}$, the corresponding $Y'(r_+)$ is always negative; on the contrary if $E_m \geqslant E_{\max}$ which is more interesting, the corresponding $Y'(r_+)$ can be non-negative in the range of $\tilde{\delta}_L \leqslant \tilde{\delta} \leqslant \tilde{\delta}_R$ and is negative elsewhere with the boundaries defined as

$$\tilde{\delta}_L = \frac{l^3 E_m r_+^2(l^2 - s^2)(lr_+ + r_- s) - \sqrt{\Delta}}{l^3(lr_- + r_+ s)\left(l\left(r_-^2 + r_+^2\right) + 2r_- r_+ s\right)},$$

$$\tilde{\delta}_R = \frac{l^3 E_m r_+^2(l^2 - s^2)(lr_+ + r_- s) + \sqrt{\Delta}}{l^3(lr_- + r_+ s)\left(l\left(r_-^2 + r_+^2\right) + 2r_- r_+ s\right)},$$

$$\Delta = l^3 r_+^2(l^2 - s^2)^2(lr_- + r_+ s)^2$$
$$\left(l^3 E_m^2 r_-^2 + m^2\left(r_-^2 - r_+^2\right)\left(l\left(r_-^2 + r_+^2\right) + 2r_- r_+ s\right)\right).$$ (50)

For the extremal black hole, $E_{\max} = 0$. It worths to note that particles with subcritical total angular momenta $J_c - \tilde{\delta}$ satisfying $\tilde{\delta}_L \leqslant \tilde{\delta} \leqslant \tilde{\delta}_R$, have $Y(r_+) > 0$ with $Y'(r_+) > 0$ and luckily, can exist outside the event horizon, which is in contrast to the non-existence of particles with critical total angular momentum in subsection IVA.

For spinless particle, the infinite center-of-mass energy collision happens at the extreme point of $Y(r)$ where usually serves as return point of particle. This is because in BTZ spacetime, except the point where the particle starts to fall, $Y(r)$ has no other zero point which is usually taken as the collision point[21]. In order to find this turning point of radial motion, we solve the equation $Y'(r) = 0$ with the positive root r_m:

$$r_m = r_+\left(1 + \frac{Y'(r_+)l^2}{2r_+ m^2}\right)^{\frac{1}{4}}.$$ (51)

Consequently whether r_m is greater than event horizon r_+ relies directly on the sign of $Y'(r_+)$ that has been analysed above. It is shown when $E_m \geqslant E_{\max}$ and $\tilde{\delta}_L \leqslant \tilde{\delta} \leqslant \tilde{\delta}_R$ satisfied, we have $Y'(r_+) \geqslant 0$ which in turn implies

$$r_m \geqslant r_+,$$ (52)

Eq.(4.19) means the turning point of radial motion is on or outside the event horizon.

After applying the extremal condition $r_- = r_+$, $E_{\max} = 0$, the boundaries (4.17) become:

$$\tilde{\delta}_L = \frac{1}{2}E_m\left(l - s - \sqrt{(l-s)^2}\right),$$

$$\tilde{\delta}_R = \frac{1}{2}E_m\left(l - s + \sqrt{(l-s)^2}\right).$$ (53)

Thus, the relation between r_m and r_+ can be summarized as below:

Since $-l < s < l$, we have $\tilde{\delta}_L = 0 \leqslant \tilde{\delta} \leqslant E_m(l - s) = \tilde{\delta}_R$ or equivalently $E_m s \leqslant J_m \leqslant E_m l$, $r_m \geqslant r_+$; for other values of $\tilde{\delta}$, $r_m < r_+$ which is disfavored by the current discussion.

In Fig. 1, $\tilde{\delta}$ has been taken as 0.01 and we compare the effective potentials of radial motion $V(r) = -Y(r)/2$ of a particle with different spins s and a subcritical total angular momentum $J_m = E_m l - \tilde{\delta}$ in an extremal BTZ spacetime, in which the minimum points mark r_m where the particle is about to return, and it is shown r_m with spins
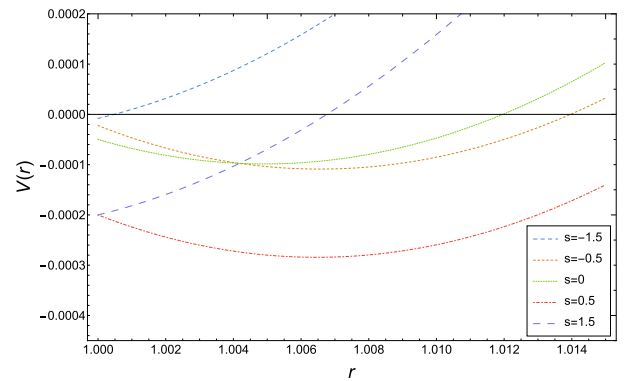

Fig. 1.   The examples of effective potential of radial motion $V(r) = -\frac{1}{2}Y(r)$ of a particle with different spins s and a subcritical total angular momentum $J_m = E_m l - \tilde{\delta} < J_c$ in an extremal BTZ spacetime. The minimum point of $V(r)$: r_m with spin $s = -0.5, 0, 0.5$ are greater than r_+. Here $r_- = r_+ = E_m = l = m = 1, \tilde{\delta} = 0.01$, and the longitudinal axis marks the event horizon r_+.

satisfying $-l < s < l$ are greater than r_+.

5   Collision of spinning particles near an extremal BTZ black hole and its critical total angular momentum limit

Note that the divergence condition for center-of-mass energy with critical total angular momentum J_c in section III was found to be unavailable in subsection IVA. Due to the timelike constraint, both spinning particles are required to possess subcritical values of total angular momentum $J_{m1} = E_{m1}l - \tilde{\delta}_1, J_{m2} \leqslant E_{m2}l$, with $\tilde{\delta}_{L1} \leqslant \tilde{\delta}_1 \leqslant \tilde{\delta}_{R1}$ because we pick r_{m1} as the collision point. Then we consider the collision center-of-mass energy E_{cm}^2 by taking the limit $\tilde{\delta}_1 \to 0$.

$$\lim_{\tilde{\delta}_1 \to 0} E_{cm}^2(r_m) = m_1^2 + m_2^2 + Q - \frac{2l^4(E_{m1}(l-s_1))J_{m2}}{r_+^2(l^2-s_1^2)(l^2-s_2^2)}, \quad (54)$$

in which

$$Q \equiv \lim_{r \to r_m} 2 \frac{W_1(r_m)W_2(r_m) - \sqrt{Y_1(r_m)Y_2(r_m)}}{g(r_m)}. \quad (55)$$

Both numerator and denominator of Q tend to zero in the limit of $\tilde{\delta}_1 \to 0$. For this reason, we express Q using L'Hopital's rule with respect to $\tilde{\delta}_1$ as

$$Q = \lim_{\tilde{\delta}_1 \to 0} \frac{W_2(r_m)}{\dot{g}(r_m)} \left(2\dot{W}_1(r_m) - \frac{\dot{Y}_1(r_m)}{\sqrt{Y_1(r_m)}} \right)$$

$$= \lim_{\tilde{\delta}_1 \to 0} \frac{W_2(r_m)}{\dot{g}(r_m)} \left(2\dot{W}_1(r_m) - 2\sqrt{\frac{\dot{W}_1(r_m)}{(l-s_1)}} \right), \quad (56)$$

with $\cdot$ indicating derivative with respect to $\tilde{\delta}_1$, in which

$$\lim_{\tilde{\delta}_1 \to 0} \dot{W}_1(r_m) = \frac{E_{m1}^2 l^4 + m_1^2 r_+^2(l+s_1)^2}{m_1^2 r_+^2(l-s_1)(l+s_1)^2}, \quad (57)$$

and after series expansion, $\dot{g}(r_m)$ becomes:

$$\lim_{\tilde{\delta}_1 \to 0} \dot{g}(r_m) = \frac{2\tilde{\delta}_1 E_{m1}^2 l^4}{m_1^4 r_+^2 \left(l^2 - s_1^2 \right)^2}. \quad (58)$$

Eventually, collecting all the above ingredients, Q can be expressed as

$$Q = \lim_{\tilde{\delta}_1 \to 0} \frac{W_2(r_m)}{\dot{g}(r_m)} \left(2\dot{W}_1(r_m) - 2\sqrt{\frac{\dot{W}_1(r_m)}{(l-s_1)}} \right)$$

$$= 2k \lim_{\tilde{\delta}_1 \to 0} \frac{W_2(r_m)}{\dot{g}(r_m)} \dot{W}_1(r_m)$$

$$= k \lim_{\tilde{\delta}_1 \to 0} W_2(r_m) \frac{m_1^2(l-s_1)\left(E_{m1}^2 l^4 + m_1^2 r_+^2(l+s_1)^2\right)}{\tilde{\delta}_1 E_{m1}^2 l^4}. \quad (59)$$

where k is

$$k = 1 - \sqrt{\frac{\dot{W}_1(r_m)}{(l-s_1)}}/\dot{W}_1(r_m) = 1 - \sqrt{\frac{m_1^2 r_+^2(l+s_1)^2}{E_{m1}^2 l^4 + m_1^2 r_+^2(l+s_1)^2}} > 0. \quad (60)$$

Therefore, it is easy to see that the collision center-of-mass energy E_{cm}^2 of the two spinning particles diverges because Q diverges at the point $r = r_m$ in the limit $\tilde{\delta}_1 \to 0$.

6   Conclusions

In this paper, we have analyzed the collision center-of-mass energy of two spinning particles near BTZ black hole. Our result shows that the center-of-mass energy of two ingoing spinning particles in the near horizon limit can be arbitrarily large if one of the particles possesses a critical angular momentum and the other has a noncritical angular momentum. However, particle with critical angular momentum can not exist outside of the horizon due to the violation of timelike constraint. Moreover, we proved that the particle with a subcritical angular momentum is allowed to exist in near neighbour of an extremal BTZ black hole and the corresponding collision center-of-mass energy of two spinning particles taking place at the point near an extremal BTZ black hole can be arbitrarily large in the $\tilde{\delta}_1 \to 0$ limit.

It should be noted that there are still many important issues that need to be investigated in the future. For example, inspired by the BSW mechanism, people found that the efficiency of extracting energy from a rotating black hole which is usually called the Penrose process can be greatly improved, especially for spinning particles [12, 13, 19, 37]. Therefore, with BSW mechanism for spinning particles in hand, studying the corresponding Penrose process becomes possible, we hope to come to this issue in the near future.

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